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Non-integrability of non-linear diffusion–convection equations in two spatial dimensions

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Abstract. It is generally accepted that integrable partial differential time evolution equations possess Lie-Bäcklund symmetries of arbitrarily high finite order. It has already been established that in $(1+1)$ dimensions the full class of non-linear Fokker-Planck (convection-diffusion) equations having this property consists of the known integrable examples. These are closely related either to the Burgers equation or to the Fokas-Yortsos-Rosen equation, both of which have been applied to unsaturated flow in porous media. Here we show that higher-order Lie-Bäcklund symmetries do not exist for any examples of the non-linear Fokker-Planck equation in $(1+2)$ (one time and two space) dimensions.

1. Introduction

For situations in which the moisture field in a non-swelling homogeneous isotropic porous medium is monotonic in time, the unsaturated flow is governed by the non-linear Fokker-Planck equation (Klute 1952, Philip 1969a):

$$\theta_t = \nabla \cdot (D(\theta)\nabla\theta) - (\mathrm{d}K/\mathrm{d}\theta)\partial\theta/\partial z, \tag{1}$$

where $\theta(t, \boldsymbol{x})$ is the volumetric moisture fraction. Laboratory measurements show that for typical soils, the diffusivity D and conductivity K may each increase by a factor of 10^3 or more as θ is varied from near dryness to near saturation. Although equation (1) is parabolic, in one-dimensional systems it admits a finite-speed travelling wave solution at large times, provided the curve for the $K(\theta)$ relationship is concave upwards (Philip 1957a), which is the case for all common soils.

It must be stressed that in this application the convective term in (1) reflects the local gravitational field and is unidimensional in character. In the absence of gravity, equation (1) would describe an isotropic diffusion process but in practical applications, the presence of one or both of gravity and multi-dimensional source geometry leads to solutions quite different from the pure one-dimensional diffusion model (Philip 1969a, b). In the presence of gravity, we lose both isotropy and additional symmetry based on similarity variables (e.g. the Boltzmann variable $zt^{-1/2}$ for one-dimensional pure diffusion). In this respect, the related soluble non-linear wave equation, in which the radial coordinate $(x^2+z^2)^{1/2}$ replaces t and the d'Alembert variable $z-ct$ replaces z in (1) (Bartucelli and Pantano 1983), is not relevant to the exacting physical problem discussed here.

In many important physical applications, unsaturated flow is essentially multi-dimensional in character (e.g. Philip 1983). However, for time evolutions of the type

(1), with convective and diffusive terms both non-linear, we know of no exact unsteady solutions in (1+2) dimensions. Therefore, we should examine the known methods of obtaining exact time-dependent solutions in (1+1) dimensions with a view to extension. Any barely relevant solvable (1+2)-dimensional system would at least provide a test for the numerical schemes currently aimed at more realistic physical situations.

In § 2 we re-examine the known techniques of exact solution. Integrable (1+1)-dimensional systems have previously been exposed using the general theory of Lie-Bäcklund symmetries. The known integrable equations possess Lie-Bäcklund symmetries of arbitrarily high finite order. Having adopted this as a general property of integrable equations, in § 4 we establish our principal result: in one time and two space dimensions, there is no integrable non-linear scalar Fokker-Planck equation.

2. Known techniques for exact solution

We assume the initial condition $\theta(0, z) = \theta_n$ in the region $z > 0$. Given concentration boundary conditions $\theta(t, 0) = \theta_0$ and $\theta \to \theta_n$ as $z \to \infty$, (1) may be solved in the (1+1)-dimensional case by the quasi-analytic method of Philip (1957b). For cylindrical or spherical symmetry, this method may be used to determine a small-t solution consisting of the first three terms in the expansion of $r(\theta)$ as a power series in $t^{1/2}$ (Philip 1969b). For large t, the moisture field may be approximated by known steady solutions (Philip 1984).

In (1+1) dimensions the best known integrable non-linear example of (1) is Burgers' equation

$$\frac{\partial \theta}{\partial t} = D\frac{\partial^2 \theta}{\partial z^2} - (a\theta + b)\frac{\partial \theta}{\partial z} \qquad D \text{ constant.} \tag{2}$$

The exact solution of (2) for the case of constant flux boundary conditions has been used by Clothier *et al* (1981) to model the infiltration of a field soil during steady rainfall. Burgers' equation may be solved by applying the Hopf-Cole transformation (Hopf 1950, Cole 1951)

$$a\theta + b = -2Du^{-1}\,\partial u/\partial z. \tag{3}$$

Then (2) becomes

$$2Da^{-1}u^{-1}\left(u^{-1}\frac{\partial u}{\partial z} - \frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial t} - D\frac{\partial^2 u}{\partial z^2}\right) = 0. \tag{4}$$

$u = \phi(z, t)$ is a solution to (4) if and only if

$$\frac{\partial \phi}{\partial t} - D\frac{\partial^2 \phi}{\partial z^2} = c(t)\phi, \tag{5}$$

for some function c of t. Generalising (3), we define a generalised Hopf-Cole transformation as a relation of the type $\theta = f(u, u_z)$, with f almost everywhere twice differentiable on $\mathbb{R}^2$. However, this definition is too restrictive since it may be proved, using the method of Nimmo and Crighton (1982), that in (1+1) dimensions only Burgers' equation may be linearised by such a device and that in (1+2) dimensions, no such linearisation of any equation of type (1) is possible (Broadbridge 1985). In fact, the more general theory of Lie-Bäcklund transformations has already been successful in exposing other solvable equations of type (1), and we should now turn in this direction.

A time evolution equation

$$u_t = K(t, z, u, u_1, \ldots, u_n), \tag{6}$$

with K a polynomial in u and its spatial derivatives u_j up to nth order, determines a flow

$$u(z, t) = \hat{K}_t u(z, 0). \tag{7}$$

A flow $v(z, t) = \hat{L}_t v(z, 0)$, determined by a time evolution equation of order m, $v_t = L(t, z, v, v_1, \ldots, v_m)$, is said to be a one-parameter symmetry group for equation (6) if the flows $\hat{K}_t$ and $\hat{L}_s$ commute,

$$\hat{L}_s[\hat{K}_t f(z)] = \hat{K}_t[\hat{L}_s f(z)]$$

for all $f \in C^\infty(\mathbb{R})$ and all s,t in some non-empty open interval in $\mathbb{R}$, containing O. In infinitesimal form the symmetry L_s is expressed

$$u^* = u + sL(t, z, u, u_1, \ldots, u_m) + O(s^2). \tag{8}$$

For example, if ϕ satisfies (5), then so does $\phi_n = (\partial/\partial z)^n \phi$. Therefore, by superposition, the linear equation (5) (and hence (4)) is left invariant by mth-order infinitesimal transformations of the type

$$\begin{aligned} &u^* = u + s\left(\sum_{j=1}^{m} a_j u_j\right) + O(s^2) \qquad a_j \in \mathbb{R} \\ &u_i^* = u_i + s\left(\sum_{j=1}^{m} a_j u_{i+j}\right) + O(s^2) \\ &t^* = t \qquad z^* = z. \end{aligned} \tag{9}$$

Transformations of the type (9), in which u_i^* depend on progressively higher-order derivatives as i increases, may be viewed as Lie-Bäcklund contact transformations on a necessarily infinite-dimensional $(t, z, u, u_1, \ldots, u_j, \ldots)$ space (Ibragimov and Anderson 1977). In the general case (6), an obvious symmetry is given by $\hat{L}_s = \hat{K}_s$. In the particular case with $K = c(t) + u_2$, from (9) one may obviously generate a chain of symmetries

$$L^{(j)} = (\partial/\partial z)^j K = u_{2+j}.$$

The symmetry recursion operator $\partial/\partial z$ is nothing other than the generator of the space translation symmetry group for equation (5),

$$u(z+s, t) = \exp(s\partial/\partial z)\, u(z, t).$$

After applying the Hopf-Cole transformation (3), this chain of elementary symmetries for the linear diffusion equation transforms to a chain of non-trivial symmetries for the Burgers' equation (Bluman and Kumei 1980, Olver 1977). The existence of Lie-Bäcklund symmetries L of arbitrarily high finite order m is perceived as a general property of integrable evolution equations (e.g. Fokas 1980). In fact, the existence of any generalised Lie-Bäcklund symmetry has proven to be a stringent requirement. For example, Bluman and Kumei (1980) have shown that if a non-linear diffusion equation

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z}\left(D(\theta)\frac{\partial \theta}{\partial z}\right) \tag{10}$$

possesses any such symmetry other than a Lie contact transformation, then D must have the form

$$D(\theta) = (\gamma + \beta\theta)^{-2} \qquad \gamma, \beta \in \mathbb{R}. \tag{11}$$

The equations (10) and (11) had already been used to obtain explicit solutions for horizontal (gravity-free) unsaturated flow by Knight and Philip (1974) by reinterpreting, with some correction, the work of Storm (1951). In the approach of Fokas (1980), one proceeds to find integrable equations after assuming the existence of a symmetry recursion operator Δ which maps z translation to t translation. Such a recursion operator exists for almost all known integrable time evolution equations. However, for the infiltration equation (1), such a recursion operator exists only if D is of the form (11) and

$$K'(\theta) = -\alpha D(\theta) \qquad \alpha > 0 \tag{12}$$

as demonstrated by Fokas and Yortsos (1982), who applied this version of equation (1) to gravity-free two-phase unsaturated flow. Subsequently, Rogers *et al* (1983) found an explicitly solvable model of the type (11) but with

$$K'(\theta) = -\alpha(\beta\theta + \gamma)^{-2}[1 + \varepsilon(\theta - \phi)][(-\alpha/\beta)(\beta\theta + \gamma)^{-1} + \delta]\varepsilon\theta, \qquad \delta, \varepsilon, \phi \in R,$$

allowing gravity to be incorporated in the model of two-phase unsaturated flow.

The system (1), (11) and (12) with constant flux boundary conditions transforms to Burgers' equation under a combination of the well known Kirchhoff (1894) and Storm (1951) transformations

$$\Theta = \int_{\theta_n}^{\theta} D \,\mathrm{d}\theta \tag{13}$$

$$z^*(t, z) = \int_0^z D^{-1/2}(\theta(t, z_1)) \,\mathrm{d}z_1 \tag{14}$$

$$\frac{\partial \Theta(t, z^*)}{\partial t} = \frac{\partial^2 \Theta}{\partial z^{*2}} - (2\alpha\beta^{-2}\Theta + \beta R)\frac{\partial \Theta}{\partial z^*}, \tag{15}$$

with R the fixed moisture flux at the surface $z = 0$.

The explicitly solvable system (1), (11) and (12) has also been noticed by Rosen (1982), who applied it to a chemical substance subject to diffusion, convection and adsorption to the walls of a porous medium. Rosen's unnamed transformations ($c \to \theta$, $x \to \hat{x}$ and $\theta \to \psi$) are essentially the Kirchhoff transformation (13), the Storm transformation (14) and the Hopf–Cole transformation (3), respectively.

It must be stressed that the above results apply only in one spatial dimension. However, the theory of Lie–Bäcklund symmetries has produced useful results and it is natural to direct the same theory at two spatial dimensions.

3. Non-integrability in two spatial dimensions

Applying the Kirchhoff transformation (13), the (1 + 2)-dimensional version of equation (1) becomes

$$\frac{\partial \Theta}{\partial t} = E(\Theta)\left(\frac{\partial^2 \Theta}{\partial x^2} + \frac{\partial^2 \Theta}{\partial z^2}\right) - P'(\Theta)\frac{\partial \Theta}{\partial z} \tag{16}$$

where

$$E(\Theta) = D(\theta(\Theta)) \qquad \text{and} \qquad P'(\Theta) = E \,\mathrm{d}K/\mathrm{d}\Theta. \tag{17a, b}$$

We shall search for infinitesimal Lie–Bäcklund symmetries of the form

$$x^* = x \qquad t^* = t$$
$$\Theta^* = \Theta + sL(x, z, t, \{\Theta_{i,j};\ i+j \leqslant n\}) + \mathrm{O}(s^2) \tag{18}$$
$$\left[\Theta_{i,j} = \left(\frac{\partial}{\partial x}\right)^i \left(\frac{\partial}{\partial z}\right)^j \Theta(t, x, z)\right].$$

The procedure adopted will largely be that used by Bluman and Kumei (1980) for the non-linear diffusion equation. Equation (18) extends to derivatives of Θ^* via

$$\Theta^*_{i,j} = (\mathrm{D}_x)^i (\mathrm{D}_z)^j (\Theta + sL) + \mathrm{O}(s^2) \tag{19}$$

where D_x and D_z are respectively the total x derivative and total z derivative;

$$\mathrm{D}_x = \frac{\partial}{\partial x} + \sum_{i,j=0}^{\infty} \Theta_{i+1,j} \frac{\partial}{\partial \Theta_{i,j}}$$

and

$$\mathrm{D}_z = \frac{\partial}{\partial z} + \sum_{i,j=0}^{\infty} \Theta_{i,j+1} \frac{\partial}{\partial \Theta_{i,j}}. \tag{20a}$$

Similarly, D_t will denote the total time derivative;

$$\mathrm{D}_t = \frac{\partial}{\partial t} + \sum_{i,j=0}^{\infty} \Theta_{i,j,t} \frac{\partial}{\partial \Theta_{i,j}}, \qquad \text{with } \Theta_{i,j,t} = \frac{\partial \Theta_{i,j}}{\partial t}. \tag{20b}$$

Since (17) is supposed to be a symmetry for equation (16),

$$\begin{aligned} 0 = \frac{\partial \Theta^*}{\partial t} &- E(\Theta^*)(\Theta^*_{2,0} + \Theta^*_{0,2}) + P'(\Theta^*)\Theta^*_{0,1} \\ &= -s\{-\mathrm{D}_t L + LE'(\Theta)(\Theta_{2,0} + \Theta_{0,2}) + E(\Theta)[(\mathrm{D}_x)^2 L + (\mathrm{D}_z)^2 L] \\ &\quad - P'(\Theta)\mathrm{D}_z L - LP''(\Theta)\Theta_{0,1}\} + \mathrm{O}(s^2) \qquad \text{(by (16) and (17)).} \end{aligned} \tag{21}$$

Now time derivatives of Θ_{ij} may be expressed solely in terms of space derivatives whenever Θ is a solution to the governing equation (16):

$$\begin{aligned} \Theta_{i,j,t} &= \left(\frac{\partial}{\partial x}\right)^i \left(\frac{\partial}{\partial z}\right)^j \frac{\partial \Theta(t, x, z)}{\partial t} \\ &= (\mathrm{D}_x)^i (\mathrm{D}_z)^j \{E(\Theta)(\Theta_{2,0} + \Theta_{0,2}) - P'(\Theta)\Theta_{0,1}\}. \end{aligned}$$

Equation (21) then implies

$$\begin{aligned} 0 = -\frac{\partial L}{\partial t} &- \sum_{i+j \leqslant n} L_{i,j} (\mathrm{D}_x)^i (\mathrm{D}_z)^j [E(\Theta)(\Theta_{2,0} + \Theta_{0,2}) - P'(\Theta)\Theta_{0,1}] + LE'(\Theta)(\Theta_{2,0} + \Theta_{0,2}) \\ &+ E(\Theta)\left[\frac{\partial^2 L}{\partial x^2} + \frac{\partial^2 L}{\partial z^2} + 2 \sum_{i+j \leqslant n} \left(\frac{\partial L_{i,j}}{\partial x} \Theta_{i+1,j} + \frac{\partial L_{i,j}}{\partial z} \Theta_{i,j+1}\right)\right. \\ &+ \sum_{\substack{i+j \leqslant n \\ k+l \leqslant n}} L_{i,j,k,l} (\Theta_{i+1,j}\Theta_{k+1,l} + \Theta_{i,j+1}\Theta_{k,l+1}) \\ &\left. + \sum_{i+j \leqslant n} L_{i,j} (\Theta_{i+2,j} + \Theta_{i,j+2})\right] \\ &- P'(\Theta)\left(\frac{\partial L}{\partial z} + \sum_{i+j \leqslant n} L_{i,j}\Theta_{i,j+1}\right) - LP''(\Theta)\Theta_{0,1}. \end{aligned} \tag{22}$$

where

$$L_{i,j} = \partial L/\partial \Theta_{i,j} \qquad \text{and} \qquad L_{i,j,k,l} = \partial^2 L/\partial \Theta_{i,j}\,\partial \Theta_{k,l}.$$

We shall show that there cannot exist Lie–Bäcklund symmetries of arbitrarily high order n. The $(n+2)$-order terms in (22) vanish identically. However, (22) remains true for an arbitrary solution $\Theta(t, x, z)$ of (16). Therefore, (22) may be viewed as a polynomial equation in the $(n+1)$th-order derivatives $\Theta_{i,n+1-i}$. Equating coefficients of second degree $\Theta_{i,n+1-i}\,\Theta_{j,n+1-j}$ terms on each side of (22), we obtain

$$\begin{aligned}
&L_{0,n,k,n-k} = 0 && k = 0, \ldots, n\\
&L_{0,n,k-1,n-k+1} + L_{1,n-1,k,n-k} = 0 && k = 1, \ldots, n\\
&L_{1,n-1,k-1,n-k+1} + L_{2,n-2,k,n-k} = 0 && k = 1, \ldots, n\\
&L_{n-1,1,k-1,n-k+1} + L_{n,0,k,n-k} = 0 && k = 1, \ldots, n\\
&L_{n,0,k-1,n-k+1} = 0 && k = 1, \ldots, n+1.
\end{aligned} \tag{23}$$

Since $L_{i,j,k,l} = L_{k,l,i,j}$, the system of equations (23) has only the trivial solution $L_{i,n-i,k,n-k} = 0$ for all $i,k = 0, \ldots, n$. That is, L is at most first degree in $\Theta_{i,n-i}$:

$$L = \sum_{i=0}^{n} [F_i(t, x, z, \{\Theta_{k,l};\, k+l \leq n-1\})\Theta_{i,n-i} + G_i(t, x, z, \{\Theta_{k,l};\, k+l \leq n-1\})], \tag{24}$$

for some differentiable functions F_i and G_i.

When we equate coefficients of first degree, $(n+1)$th-order $\Theta_{i,n+1-i}$ terms in (22), we now obtain

$$2E(\Theta)\left(\sum_{k+l\leq n-1} L_{0,n,k,l}\Theta_{k,l+1} + \frac{\partial L_{0,n}}{\partial z}\right) - E'(\Theta)\{n\Theta_{0,1}L_{0,n} + \Theta_{1,0}L_{1,n-1}\} = 0 \tag{25a}$$

$$2E(\Theta)\left(\sum_{k+l\leq N-1} L_{n,0,k,l}\Theta_{k+1,l} + \frac{\partial L_{n,0}}{\partial x}\right) - E'(\Theta)(\Theta_{0,1}L_{n-1,1} + n\Theta_{1,0}L_{n,0}) = 0 \tag{25b}$$

$$\begin{aligned}
2E(\Theta)\Bigg(&\sum_{k+l\leq n-1} L_{1,n-1,k,l}\Theta_{k,l+1}\\
&+ \sum_{k+l\leq n-1} L_{0,n,k,l}\Theta_{k+1,l} + \frac{\partial L_{1,n-1}}{\partial z} + \frac{\partial L_{0,n}}{\partial x}\Bigg)\\
&-E'(\Theta)[(n-1)\Theta_{0,1}L_{1,n-1} + 2\Theta_{1,0}L_{2,n-2}] = 0
\end{aligned} \tag{25c}$$

$$\begin{aligned}
2E(\Theta)\Bigg(&\sum_{k+l\leq n-1} L_{n,0,k,l}\Theta_{k,l+1} + \sum_{k+l\leq n-1} L_{n-1,1,k,l}\Theta_{k+1,l} + \frac{\partial L_{n,0}}{\partial z} + \frac{\partial L_{n-1,1}}{\partial x}\Bigg)\\
&- E'(\Theta)[(n-1)\Theta_{1,0}L_{n-1,1} + 2\Theta_{0,1}L_{n-2,2}] = 0
\end{aligned} \tag{25d}$$

$2 \leq i \leq n-1$:

$$\begin{aligned}
2E(\Theta)\Bigg(&\sum_{k+l\leq n-1} L_{i,n-i,k,l}\Theta_{k,l+1} + \sum_{k+l\leq n-1} L_{i-1,n+1-i,k,l}\Theta_{k+1,l} + \frac{\partial L_{i,n-1}}{\partial z} + \frac{\partial L_{i-1,n-i+1}}{\partial x}\Bigg)\\
&-E'(\Theta)[(n-i+2)\Theta_{0,1}L_{i-2,n-i+2} + (i-1)\Theta_{1,0}L_{i-1,n-i+1}\\
&+(n-i)\Theta_{0,1}L_{i,n-i} + (i+1)\Theta_{1,0}L_{i+1,n-1-i}] = 0.
\end{aligned} \tag{25e}$$

Equating coefficients of nth-order $\Theta_{k,n-k}$ terms in (25), and using (24), we obtain

$$\frac{\partial F_n}{\partial \Theta_{k,n-1-k}} = \frac{\partial F_0}{\partial \Theta_{k,n-1-k}} = 0, \qquad k = 0, \ldots, n-1$$

$$\frac{\partial F_i}{\partial \Theta_{n-1,0}} = \frac{\partial F_i}{\partial \Theta_{0,n-1}} = 0, \qquad i = 1, \ldots, n-1 \tag{26}$$

$$\frac{\partial F_i}{\partial \Theta_{k,n-1-k}} + \frac{\partial F_{i-1}}{\partial \Theta_{k-1,n-k}} = 0, \qquad i = 1, \ldots, n-1, \qquad k = 1, \ldots, n-1.$$

The system of equations (26) is equivalent to

$$\partial F_i / \partial \Theta_{k,n-1-k} = 0, \qquad i = 0, \ldots, n, \qquad k = 0, \ldots, n-1.$$

Therefore, summations $\Sigma_{k+l \leqslant n-1}$ in (25) may be replaced by $\Sigma_{k+l \leqslant n-2}$. Equating coefficients of $(n-1)$th-order $\Theta_{k,n-k-1}$ terms in (25) and using (24), the same argument as before shows that F_i does not depend on $\Theta_{k,n-2-k}$. We can show progressively that F_i does not depend on $n-1, n-2, n-3, \ldots$, first-order derivatives of Θ. Therefore,

$$F_i = F_i(t, x, z, \Theta). \tag{27}$$

In the following, we shall first assume $E'(\Theta) \neq 0$ (case I).

Equating coefficients of $\Theta_{1,0}$ terms in (25), we obtain

$$F_1 = 0 \tag{28a}$$

$$E(\Theta) \frac{\partial F_0}{\partial \Theta} - E'(\Theta) F_2 = 0 \tag{28b}$$

$$(i+1) E'(\Theta) F_{i+1} = 2E(\Theta) \frac{\partial F_{i-1}}{\partial \Theta} - (i-1) E'(\Theta) F_{i-1}, \qquad i = 2, \ldots, n-1 \tag{28c}$$

$$\frac{2}{n-1} E(\Theta) \frac{\partial F_{n-1}}{\partial \Theta} - E'(\Theta) F_{n-1} = 0. \tag{28d}$$

Now we equate coefficients of $\Theta_{0,1}$ terms in (25) and obtain

$$F_{n-1} = 0 \tag{29a}$$

$$\frac{2}{n-1} E(\Theta) \frac{\partial F_1}{\partial \Theta} - E'(\Theta) F_1 = 0 \tag{29b}$$

$$(n-i+2) E'(\Theta) F_{i-2} = 2E(\Theta) \frac{\partial F_i}{\partial \Theta} - (n-i) E'(\Theta) F_i \qquad \text{for } 2 \leqslant i \leqslant n-1 \tag{29c}$$

$$E(\Theta) \frac{\partial F_n}{\partial \Theta} - E'(\Theta) F_{n-2} = 0 \tag{29d}$$

$$2E(\Theta) \frac{\partial F_0}{\partial \Theta} - nE'(\Theta) F_0 = 0. \tag{29e}$$

Equations (28)-(29) have the general solution

$$F_{2j+1} = 0$$
$$F_{2j} = \binom{n/2}{j} g(t, x, z) E(\Theta)^{n/2}, \tag{30}$$

for some function g. That is,

$$L = g(t, x, z)E(\Theta)^{n/2} \sum_{i=0}^{n/2} \binom{n/2}{i} \Theta_{2i,n-2i} + G(t, x, z)\{\Theta_{i,j};\ i+j \leq n-1\}, \tag{31}$$

for some function G of order less than n.

Balancing nth-order terms in (22), we obtain

$$\begin{aligned}
0 = -gE^{n/2}\Bigg[&(E''\Theta_{0,1}^2 + E'\Theta_{0,2}) \sum_{i=0}^{n/2-1} \binom{n/2}{i}\binom{n-2i}{2}(\Theta_{2i+2,n-2i-2} + \Theta_{2i,n-2i}) \\
&+ (E''\Theta_{1,0}^2 + E'\Theta_{2,0}) \sum_{i=1}^{n/2} \binom{n/2}{i}\binom{2i}{2}(\Theta_{2i,n-2i} + \Theta_{2i-2,n-2i+2}) \\
&+ (E''\Theta_{1,0}\Theta_{0,1} + E'\Theta_{1,1}) \sum_{i=1}^{n/2-1} \binom{n/2}{i} \\
&\times 2i(n-2i)(\Theta_{2i+1,n-2i-1} + \Theta_{2i-1,n-2i+1}) \\
&- P'' \sum_{i=1}^{n/2} \binom{n/2}{i} 2i\Theta_{1,0}\Theta_{2i-1,n-2i+1} - P'' \sum_{i=0}^{n/2-1} \binom{n/2}{i}(n-2i)\Theta_{0,1}\Theta_{2i,n-2i}\Bigg] \\
&- E'\Theta_{1,0} \sum_{i=1}^{n-1} G_{i,n-1-i}\, i(\Theta_{i+1,n-1-i} + \Theta_{i-1,n+1-i}) \\
&- E'\Theta_{0,1} \sum_{i=0}^{n-2} G_{i,n-1-i}(n-1-i)(\Theta_{i+2,n-2-i} + \Theta_{i,n-i}) \\
&- \frac{\partial g}{\partial t} E^{n/2} \sum_{i=0}^{n/2} \binom{n/2}{i} \Theta_{2i,n-2i} \\
&+ nE'E^{n/2}\left(\Theta_{1,0}\frac{\partial g}{\partial x} + \Theta_{0,1}\frac{\partial g}{\partial z}\right)\left[\sum_{i=0}^{n/2} \binom{n/2}{i} \Theta_{2i,n-2i}\right] \\
&+ E^{n/2+1}\left(\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial z^2}\right) \sum_{i=0}^{n/2} \binom{n/2}{i} \Theta_{2i,n-2i} \\
&+ 2E \sum_{i=0}^{n-1} \left(\frac{\partial^2 G}{\partial\Theta_{i,n-1-i}\partial x}\Theta_{i+1,n-1-i} + \frac{\partial^2 G}{\partial\Theta_{i,n-1-i}\partial z}\Theta_{i,n-i}\right) \\
&+ \frac{n}{2} gE^{n/2-1}\left[\left(\frac{n}{2}-1\right)(E')^2 + EE''\right](\Theta_{1,0}^2 + \Theta_{0,1}^2) \sum_{i=0}^{n/2} \binom{n/2}{i} \Theta_{2i,n-2i} \\
&+ E \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} \frac{\partial^2 G}{\partial\Theta_{i,n-1-i}\partial\Theta_{j,n-1-j}}(\Theta_{i+1,n-1-i}\Theta_{j+1,n-1-j} + \Theta_{i,n-i}\Theta_{j,n-j}) \\
&+ 2E \sum_{i=0}^{n-1} \sum_{l+j\leq n-2} \frac{\partial^2 G}{\partial\Theta_{i,n-1-i}\partial\Theta_{j,l}}(\Theta_{i+1,n-1+i}\Theta_{j+1,l}\Theta_{i,n-i}\Theta_{j,l+1}) \\
&- P'(\Theta)E^{n/2}\frac{\partial g}{\partial z} \sum_{i=0}^{n/2} \binom{n/2}{i} \Theta_{2i,n-2i}.
\end{aligned} \tag{32}$$

By considering second degree nth-order $\Theta_{i,n-i}\Theta_{j,n-j}$ terms in (32), we obtain a set of equations for G analogous to equations (23) for L, implying that G is at most first degree in $(n-1)$th-order terms. That is,

$$L=gE^{n/2}\sum_{i=0}^{n/2}\binom{n/2}{i}\Theta_{2i,n-2i}+\sum_{i=0}^{n-1}H_i(t,x,z,\{\Theta_{i,j;i+j\le n-2}\})\Theta_{i,n-1-i}$$
$$+J(t,x,z,\{\Theta_{i,j;i+j\le n-2}\}) \tag{33}$$

for some functions H_i and J.

By equating coefficients of nth-order$\times(n-1)$th-order $\Theta_{i,n-i}\Theta_{j,n-1-j}$ terms in (32), we deduce that the functions H_i are independent of $(n-2)$th-order derivatives $\Theta_{i,n-2-i}$. Similarly, by equating consecutively coefficients of nth order$\times m$th order ($m=n-2, n-3,\ldots,3$), we find that H_i are of the first order at most. Now by equating the coefficients of $\Theta_{0,n}\Theta_{0,2}$ in (32), we obtain

$$\partial H_0/\partial\Theta_{0,1}=\tfrac{1}{4}n(n-1)gE'(\Theta)E^{n/2-1}. \tag{34a}$$

If we had carried through this analysis of (1) in the original dependent variable θ, then the unmasked $D'(\theta)\nabla\theta\nabla\theta$ term in (1) would have contributed $-ng\, D^{n/2-1}D'(\theta)$ to the left-hand side of (34a). Hence, (34a) is equivalent to

$$\partial H_0/\partial\theta_{0,1}=\tfrac{1}{4}n(n+3)gD'(\Theta)D^{n/2-1}. \tag{34b}$$

Equation (34b) is already known from the one-dimensional case (equation (14) of Bluman and Kumei (1980)). However, in the two-dimensional case, we need also to balance cross products of x derivatives by z derivatives. By equating coefficients of $\Theta_{0,n}\Theta_{2,0}$ in (32), we immediately obtain $E'(\Theta)=0$. Unlike the one-dimensional case, there cannot exist high order ($n\ge 4$) Lie-Bäcklund symmetries for any non-linear diffusion equation in $(1+2)$ dimensions.

It now remains to address case II, in which $E'(\Theta)=0$. We have not yet ruled out the possibility of a $(1+2)$-dimensional integrable Fokker-Planck equation which, like the $(1+1)$-dimensional Burgers' equation, contains a linear diffusive term and a non-linear convective term. As in case I, steps down to and including equation (27) remain valid. If, as before, we then equate coefficients of $\Theta_{1,0}$ terms in (25), we obtain $\partial F_i/\partial\Theta=0$. That is,

$$F_i=F_i(t,x,z),\qquad \text{for } i=0,\ldots,n. \tag{35}$$

Equations (25) then simplify to

$$\partial F_n/\partial x=\partial F_0/\partial z=0 \tag{36a}$$

$$\partial F_i/\partial z+\partial F_{i-1}/\partial x=0,\qquad i=1,\ldots,n. \tag{36b}$$

Now from the nth-order $\Theta_{i,n-i}$ terms in (22), we obtain

$$-\sum_{i=0}^{n}\frac{\partial F_i}{\partial t}\Theta_{i,n-i}+\sum_{i=0}^{n}F_i[(n-i-1)P''(\Theta)\Theta_{0,1}\Theta_{i,n-i}+iP''(\Theta)\Theta_{1,0}\Theta_{i-1,n-i+1}]$$
$$+E\sum_{i=0}^{n}(\nabla^2F_i)\Theta_{i,n-1}+2E\sum_{i=0}^{n-1}\left(\frac{\partial^2G}{\partial x\partial\Theta_{i,n-1-i}}\Theta_{i+1,n-1-i}+\frac{\partial^2G}{\partial z\partial\Theta_{i,n-1-i}}\Theta_{i,n-i}\right)$$
$$+E\sum_{i=0}^{n-1}\sum_{k=0}^{n-1}\frac{\partial^2G}{\partial\Theta_{i,n-1-i}\partial\Theta_{k,n-1-k}}(\Theta_{i+1,n-1-i}\Theta_{k+1,n-1-k}+\Theta_{i,n-i}\Theta_{k,n-k})$$
$$+2E\sum_{i=0}^{n-1}\sum_{k+l\le n-2}\frac{\partial^2G}{\partial\Theta_{i,n-1-i}\partial\Theta_{k,l}}(\Theta_{i+1,n-1-i}\Theta_{k+1,l}+\Theta_{i,n-i}\Theta_{k,l+1})$$
$$-P'(\Theta)\sum_{i=0}^{n}\frac{\partial F_i}{\partial z}\Theta_{i,n-i}=0. \tag{37}$$

As before, by equating coefficients of second-degree nth-order $\Theta_{i,n-i}\Theta_{k,n-k}$ terms in (37), we deduce that G is at most first degree in $(n-1)$th-order derivatives $\Theta_{i,n-1-i}$:

$$
\begin{aligned}
L &= \sum_{i=0}^{n} F_i(t,x,z)\Theta_{i,n-i} + G(t,x,z,\{\Theta_{k,l};\ k+l\leqslant n-1\}) \\
&= \sum_{i=0}^{n} F_i\Theta_{i,n-i} + \sum_{i=0}^{n-1} H_i(t,x,z,\{\Theta_{k,l};\ k+l\leqslant n-2\})\Theta_{i,n-1-i} \qquad (38)\\
&\quad + J(t,x,z,\{\Theta_{k,l};\ k+l\leqslant n-2\})
\end{aligned}
$$

for some differentiable functions J and H_i.

Equating coefficients of first-degree nth-order $\Theta_{i,n-i}$ terms in (37), we obtain

$$
\begin{aligned}
&-\frac{\partial F_0}{\partial t} + (n-1)F_0P''(\Theta)\Theta_{0,1} + F_1P''(\Theta)\Theta_{1,0} + E\nabla^2F_0 + 2E\frac{\partial H_0}{\partial z} \\
&\qquad + 2E\sum_{k+l\leqslant n-2}\frac{\partial H_0}{\partial\Theta_{k,l}}\Theta_{k,l+1} - P'(\Theta)\frac{\partial F_0}{\partial z} = 0 \qquad (39a)
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{\partial F_i}{\partial t} + F_i(n-i-1)P''(\Theta)\Theta_{0,1} + (i+1)P''(\Theta)\Theta_{1,0}F_{i+1} + E\nabla^2F_i \\
&\qquad + 2E\frac{\partial H_{i-1}}{\partial x} + 2E\frac{\partial H_i}{\partial z} + 2E\sum_{k+l\leqslant n-2}\frac{\partial H_{i-1}}{\partial\Theta_{k,l}}\Theta_{k+1,l} \\
&\qquad + 2E\sum_{k+l\leqslant n-2}\frac{\partial H_i}{\partial\Theta_{k,l}}\Theta_{k,l+1} - P'(\Theta)\frac{\partial F_i}{\partial z} = 0 \qquad 1\leqslant i\leqslant n-1 \qquad (39b)
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{\partial F_n}{\partial t} + E\nabla^2F_n + 2E\frac{\partial H_{n-1}}{\partial x} + 2E\sum_{k+l\leqslant n-2}\frac{\partial H_{n-1}}{\partial\Theta_{k,l}}\Theta_{k+1,l} \\
&\qquad - P'(\Theta)\frac{\partial F_n}{\partial z} - F_nP''\Theta_{0,1} = 0. \qquad (39c)
\end{aligned}
$$

Then equating coefficients of $(n-1)$th-order terms in (39), we obtain

$$0 = \frac{\partial H_0}{\partial\Theta_{k,n-2-k}} = \frac{\partial H_{n-1}}{\partial\Theta_{k,n-2-k}} \qquad k=0,\ldots,n-2 \qquad (40a)$$

$$0 = \frac{\partial H_i}{\partial\Theta_{0,n-2}} \qquad i=1,\ldots,n-1 \qquad (40b)$$

$$0 = \frac{\partial H_i}{\partial\Theta_{n-2,0}} \qquad i=0,\ldots,n-2 \qquad (40c)$$

$$0 = \frac{\partial H_{i-1}}{\partial\Theta_{k-1,n-1-k}} + \frac{\partial H_i}{\partial\Theta_{k,n-2-k}} \qquad i=1,\ldots,n-2; \qquad k=1,\ldots,n-2, \qquad (40d)$$

which implies that the functions H_i are independent of $(n-2)$th-order derivatives $\Theta_{j,n-2-j}$. Similarly, by consecutively equating coefficients of terms of order $(n-2)$, $(n-3),\ldots,2$ in (39), we deduce that the functions H_i do not depend on derivatives $\Theta_{k,l}$

of order $(n-3), (n-4), \ldots, 1$. That is,

$$H_i = H_i(t, x, z, \Theta). \tag{41}$$

Now equating coefficients of $\Theta_{1,0}$ terms in (39), we obtain

$$p''(\Theta)F_1 = 0 \Rightarrow F_1 = 0 \tag{42a}$$

(otherwise, the governing equation (16) reduces to the linear case)

$$(i+1)P''(\Theta)F_{i+1} + 2E\,\partial H_{i-1}/\partial\Theta = 0, \qquad i = 1, \ldots, n-1 \tag{42b}$$

$$\partial H_{n-1}/\partial\Theta = 0. \tag{42c}$$

Equating coefficients of $\Theta_{0,1}$ terms in (39), we obtain

$$0 = (n-j-1)F_jP''(\Theta) + 2E\,\partial H_j/\partial\Theta, \qquad j = 0, \ldots, n-1 \tag{43a}$$

$$F_n = 0. \tag{43b}$$

From (42a, b) and (43a), it follows that

$$\partial H_j/\partial\Theta = F_j = 0 \qquad \forall \text{ odd } j. \tag{44}$$

Consequently, by (36), $F_j = F_j(t)$ for all even j.

If n is even, then it follows from (42b, c) and (43) that F_j vanishes also for all even j and hence L does not depend on nth-order derivatives $\Theta_{i,n-i}$. There remains the case that n is odd. From (42) and (43), we then have

$$F_{2j} = \binom{\frac{1}{2}(n-1)}{j}F_0(t) \tag{45a}$$

$$\frac{\partial H_{2j}}{\partial\Theta} = -\frac{P''(\Theta)}{2E}(n-2j-1)\binom{\frac{1}{2}(n-1)}{j}F_0(t) \qquad \text{for } j = 1, 2, \ldots, (n-1)/2. \tag{45b}$$

We may identify the $(n-1)$th-order terms in (22), after applying the repeated binomial expansion

$$(\mathrm{D}_x)^i(\mathrm{D}_z)^j[P'(\Theta)\Theta_{0,1}] = \sum_{k=0}^{i}\sum_{l=0}^{j}\binom{i}{k}\binom{j}{l}\Theta_{i-k,j+1-l}(\mathrm{D}_x)^k(\mathrm{D}_z)^lP'(\Theta).$$

From the $(n-1)$th-order $\Theta_{i,n-1-i}$ terms in (22), we obtain

$$\begin{aligned}
-\sum_{i=0}^{n-1}\frac{\partial H_i}{\partial t}\Theta_{i,n-1-i} &+ \sum_{i=0}^{\frac{1}{2}(n-1)}F_{2i}(t)\Theta_{0,1}\\
&\times[2iP'''(\Theta)\Theta_{1,0}\Theta_{2i-1,n-2i} + (n-2i)P'''(\Theta)\Theta_{0,1}\Theta_{2i,n-2i-1}]\\
&+ nF_nP'''(\Theta)\Theta_{1,0}\Theta_{0,1}\Theta_{n-1,0}\\
&+ \sum_{i=0}^{n-1}H_i[P''(\Theta)\Theta_{0,1}\Theta_{i,n-1-i} + iP''(\Theta)\Theta_{1,0}\Theta_{i-1,n-i}
\end{aligned}$$

$$
\begin{aligned}
&+(n-1-i)P''(\Theta)\Theta_{0,1}\Theta_{i,n-1-i}] \\
&+\sum_{i=0}^{1/2(n-3)} F_{2i}\binom{n-2i}{2}(P'''(\Theta)\Theta_{0,1}{}^2+P''(\Theta)\Theta_{0,2})\Theta_{2i,n-1-2i} \\
&+\sum_{i=1}^{1/2(n-1)} F_{2i}(n-2i)2i(P'''(\Theta)\Theta_{1,0}\Theta_{0,1}+P''(\Theta)\Theta_{1,1})\Theta_{2i-1,n-2i} \\
&+\sum_{i=1}^{1/2(n-1)} F_{2i}\binom{2i}{2}(P'''(\Theta)\Theta_{1,0}{}^2+P''(\Theta)\Theta_{2,0})\Theta_{2i-2,n+1-2i} \\
&+E\Bigg[\sum_{i=0}^{n-1}(\nabla^2 H_i)\Theta_{i,n-1-i} \\
&+2\sum_{i=0}^{n-2}\left(\frac{\partial^2 J}{\partial x\partial\Theta_{i,n-2-i}}\Theta_{i+1,n-2-i}+\frac{\partial^2 J}{\partial z\partial\Theta_{i,n-2-i}}\Theta_{i,n-1-i}\right) \\
&+\sum_{i,k=0}^{n-2}\frac{\partial^2 J}{\partial\Theta_{i,n-2-i}\partial\Theta_{k,n-2-k}}(\Theta_{i+1,n-2-i}\Theta_{k+1,n-2-k}+\Theta_{i,n-1-i}\Theta_{k,n-i-k}) \\
&+2\sum_{i=0}^{n-2}\sum_{k+l\leqslant n-3}\frac{\partial^2 J}{\partial\Theta_{i,n-2-i}\partial\Theta_{k,l}}(\Theta_{i+1,n-2-i}\Theta_{k+1,l}+\Theta_{i,n-1-i}\Theta_{k,l+1})\Bigg] \\
&-P'(\Theta)\sum_{i=1}^{n-1}\frac{\partial H_i}{\partial z}\Theta_{i,n-1-i}-P''(\Theta)\Theta_{0,1}\sum_{i=0}^{n-1}H_i\Theta_{i,n-1-i}=0. \qquad (46)
\end{aligned}
$$

Just as we previously deduced that $\partial G/\partial\Theta_{i,n-1-i}(=H_i)$ depends only on t, x, z, and Θ, we may now deduce that $\partial J/\partial\Theta_{i,n-2-i}$ depends only on $t, x, z, \Theta, \Theta_{0,1}$ and $\Theta_{1,0}$. Therefore, we may identify $(n-1)$th-order$\times$second-order terms in (22). These are contained in

$$
\begin{aligned}
-L_{0,0}[E(\Theta_{2,0}+\Theta_{0,2})-P'(\Theta)\Theta_{0,1}]-\sum_{i=0}^{n}F_i(\mathrm{D}_x)^i(\mathrm{D}_z)^{n-i} \\
\times[E(\Theta_{2,0}+\Theta_{0,2})-P'(\Theta)\Theta_{0,1}]+EL_{0,0}(\Theta_{2,0}+\Theta_{0,2}).
\end{aligned}
$$

We may express F_i in terms of F_0, using (45a). Since the coefficients of, for example, $\Theta_{i,n-i-1}\Theta_{0,2}$ terms in (22) must be equated to zero, it follows that $F_0=0$. Hence, by (44) and (45a) $F_j=0$ for all j. Therefore, since by (24) L does not depend on $\Theta_{i,n-i}$; there exists no Lie-Bäcklund symmetry of order n. We conclude that no non-linear equation of the type (16) can have Lie-Bäcklund symmetries of arbitrarily high finite order.

4. Conclusion

This investigation of equation (16) was motivated by its application to unsaturated flow in two dimensions. We have shown that no such equation is integrable in the conventional sense. This does not rule out the possible integrability of equations more general than (1). For example, the Burgers' equation lies at the basement of a hierarchy of integrable higher-order non-linear equations in $(1+1)$ dimensions (Rogers and Sachdev 1984).

In this work we have neglected discrete, as opposed to continuous, symmetry groups. This may help to explain the exact solvability of some equations which do not possess Lie-Bäcklund symmetries. For example, the $(1+1)$-dimensional non-linear diffusion equation with $D(\theta)=\theta^{-1}$ may be solved for certain boundary conditions (Fujita 1952) even though Lie-Bäcklund symmetries cannot exist unless $D=a(b-\theta)^{-2}$ for some fixed a and b (Bluman and Kumei 1980). Finally, we must admit that even after centuries of dynamical systems theory, the exact relationship between practical solvability and integrability is still open to further study. Even if a solution may be expressed in terms of familiar functions, the latter still require computation. The relative ease of practical solution of dynamical systems may ultimately be expressed in an information theoretic sense (Eckhardt *et al* 1984).

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